*International Journal of Theoretical Physics, Vol. 42, No. 7, July 2003 (© 2003)*

# Generalized Sasaki Projections and Riesz Ideals in Pseudoeffect Algebras

**Sylvia Pulmannová**[1]

*Received March 25, 2003*

A noncommutative version of generalized Sasaki projections in pseudoeffect algebras is introduced. It is proved that an ideal in a pseudoeffect algebra is Riesz if and only if it is closed under the right and left Sasaki projections. In lattice ordered pseudoeffect algebras, it is shown that generalized Sasaki projections are one-element sets, and their explicit form is found. It is shown that if a supremum of a normal Riesz ideal in a lattice ordered pseudoeffect algebra exists, it is a central element. These results extend those obtained recently by Avallone and Vitolo for effect algebras.

**KEY WORDS:** pseudo-effect algebra; generalized Sasaki projection; Riesz ideal; central element.

## 1. INTRODUCTION

Effect algebras (alias difference posets) have been introduced for modelling unsharp measurements in quantum mechanical systems (Foulis and Bennett, 1994). They are a generlization of many structures which arose in quantum mechanics (Beltrametti and Cassinelli, 1981; Pták and Pulmannová, 1981; Varadarajan, 1985), in particular of orthomodular lattices in noncommutative measure theory and MV-algebras in fuzzy measure theory. After 1994, a great number of papers concerning effect algebras have been published (see Dvurečenskij and Pulmannová, 2000, for basic properties and bibliography).

At the end of 90s, a noncommutative version of MV-algebras (Chang, 1958), called pseudo-MV-algebras appeared (Georgescu and Iorgulescu, 2001; Rachůnek, 2002). A generalization of these, a noncommutative version of effect algebras, so-called pseudoeffect algebras, have been introduced and studied in Dvurečenskij and Vetterlein (2001a,b,c) and Dvurečenskij (2003). Noncommutative algebraic strcutures found applications in noncommutative logic (Hájek, submitted) and a programming language (Baudot, 2000).

[1] Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, 814 73 Bratislava, Slovakia; e-mail: pulmann@mat.savba.sk.



0020-7748/03/0700-1413/0 © 2003 Plenum Publishing Corporation

It has been proved in Dvurečenskij and Vetterlein (2001c) that the quotient of an effect algebra with respect to a normal Riesz ideal is again a pseudoeffect algebra. In the present paper, we introduce a noncommutative version of generalized Sasaki projections (Bennett and Foulis, 1998) in pseudoeffect algebras and show their relations to Riesz ideals. The results extend those in Chevalier and Pulmannová (2000) and Avallone and Vitolo (2003) obtained for effect algebras.

## 2. PSEUDOEFFECT ALGEBRAS

A partial algebra $(E; +, 0, 1)$, where $+$ is a partial binary operation and 0 and 1 are constants, is called a *pseudoeffect algebra*, if, for all $a, b, c \in E$, the following properties hold:

(i) $a + b$ and $(a + b) + c$ exist iff $b + c$ and $a + (b + c)$ exist, and in this case $(a + b) + c = a + (b + c)$,
(ii) there is exactly one $d \in E$ and exactly one $e \in E$ such that $a + d = e + a = 1$,
(iii) if $a + b$ exists, there are elements $d, e$ in $E$ such that $a + b = d + a = b + e$,
(iv) if $1 + a$ or $a + 1$ exists, then $a = 0$.

Define $a \leq b$ iff there exists an element $c \in E$ such that $a + c = b$. Then $\leq$ is a partial ordering on E such that $0 \leq a \leq 1$ for all $a \in E$. It can be shown that $a \leq b$ iff $b = a + c = d + a$ for some $c, d \in E$. We will write $c = a/b$ and $d = b\backslash a$. Then

$$(b\backslash a) + a = a + (a/b) = b,$$

in particular, we denote $\bar{a} = 1\backslash a$, $\tilde{a} = a/1$, so that $\bar{a} + a = 1 = a + \tilde{a}$ for all $a \in E$.

Notice that if $a + b$ is defined, and $a_1 \leq a$, $b_1 \leq b$, then $a_1 + b_1$ is also defined in $E$. In what follows, we often write $a + b$ tacitly assuming that $a + b$ is defined.

For basic properties of pseudoeffect algebras see Dvurečenskij and Vetterlein (2001a,b). We note that if $+$ is commutative, then $E$ becomes an effect algebra.

For example, if $(G, u)$ is a po-group with a strong unit $u$ (not necessarily abelian), and

$$\Gamma(G, u) := \{g \in G : 0 \leq g \leq u\}$$

is the initial interval in $G^+$, then $(\Gamma(G, u); +, 0, u)$ is a pseudoeffect algebra if we restrict the group addition $+$ to $\Gamma(G, u)$.

Let $E, F$ be two pseudoeffect algebras. A mapping $h : E \to F$ is said to be a *(homo)morphism* if

(i) $h(0) = 0$ and $h(1) = 1$ and
(ii) $h(a + b) = h(a) + h(b)$ whenever $a + b$ is defined in $E$.

If $h$ is injective and surjective, and $h^{-1}$ is a homomorphism, then $h$ is said to be an *isomorphism* and $E$ and $F$ are called *isomorphic*.

A nonempty subset $I$ of a pseudoeffect algebra $E$ is said to be an *ideal* if

(i) $x + y \in I$ whenever $x, y \in I$ and $x + y$ is defined in $E$, and
(ii) if $x \in I$ and $y \leq x$, then $y \in I$.

Clearly, $\{0\}$ and $E$ are ideals in $E$. An ideal $J$ is called *proper* if $J \neq E$.

Let $A$ be a nonempty subset of $E$, $a \in E$. Then $A + a := \{x + a: x \in A$ and $x + a$ is defined in $E\}$ and $a + A := \{a + x: x \in A$ and $a + x$ is defined in $E\}$.

An ideal in $I \subseteq E$ is called *normal* if $a + I = I + a$ for all $a \in E$. Clearly, $\{0\}$ and $E$ are normal ideals. Moreover, if $f$ is a homomorphism from $E$ to $F$, then

$$\ker(f) := \{x \in E:\ f(x) = 0\}$$

is a normal ideal of $E$.

An ideal $I$ of $E$ is said to be a *Riesz ideal* (cf. Chevalier and Pulmannová, 2000; Dvurečenskij and Vetterlein, 2001c) if

$$x \in I, a, b, \in E, x \leq a + b \ \Rightarrow\ \exists a_1, b_1 \in I, x \leq a_1 + b_1, a_1 \leq a, b_1 \leq b.$$

Let $P$ be an ideal of a pseudoeffect algebra $E$. For $a, b \in E$, we write $a \sim_I b$ iff there are elements $e, f \in I$ such that $a \backslash e = b \backslash f$. We recall that $a \sim_I b$ iff $e'/a = b\backslash f$ for some $e', f \in I$ iff $e'/a = f'/b$ for some $e', f' \in I$.

**Theorem 2.1.** (Dvurečenskij and Vetterlein, 2001c, Proposition 3.6). *Let $P$ be a normal Riesz ideal of a pseudoeffect algebra $E$. Then $\sim_p$ is an equivalence on $E$ such that $(E/P; +, [0]_P, [1]_P)$ is a pseudoeffect algebra, where $[a]_P = \{b \in E: b \sim_P a\}$, $E/P = \{[a]: a \in E\}$, and $[a]_P + [b]_P = [c]_P$ iff there are $a_1 \in [a]_P$, $b_1 \in [b]_P$, and $c_1 \in [c]_P$ such that $a_1 + b_1 = c_1$.*

## 3. GENERALIZED SASAKI PROJECTIONS

In analogy with Bennett and Foulis (1998), let us introduce the following sets:

$$\Delta_l(a, b) := \{d \in E : d \leq a, b \leq d + \tilde{a}\}, \tag{1}$$

$$\mathbf{\Phi}_l(a, b) := \{\text{minimal elements in } \nabla_l(a, b)\}; \tag{2}$$

$$\nabla_r(a, b) := \{d \in E : d \leq a, b \leq \overline{a} + d\}, \tag{3}$$

$$\mathbf{\Phi}_r(a, b) := \{\text{minimal elements in } \nabla_r(a, b)\}. \tag{4}$$

The sets $\Phi_l(a, b)$ (resp. $\Phi_r(a, b)$) will be called the *left* (resp. the *right*) *generalized Sasaki projection of $b$ on $a$*. Notice that $a \in \nabla_l(a, b)$ and $a \in \nabla_r(a, b)$, but $\Phi_l(a, b)$ and $\Phi_r(a, b)$ may be empty in general (Bennett and Foulis, 1998).

We will say that a subset $J$ of $E$ is *closed* under left or right generalized Sasaki projections if

$$j \in J, a \in E \quad \Rightarrow \quad \Phi_l(a, j) \subset J,$$

or

$$j \in J, a \in E \quad \Rightarrow \quad \Phi_r(a, j) \subset J.$$

We will say that $J$ is *closed under generalized Sasaki projections* if it is closed under left and right genealized Sassaki projections.

**Lemma 3.1.** *Let $a, b \in E$, if $s \in E$ is such that $\tilde{a}\backslash s$ and $b + s$ are defined, then*

*(i)* $\nabla_l(a, b + s) \subset \Delta_l(s' + a, b)$, *where* $\tilde{a}\backslash s = (s' + a)^{\sim}$.
*(ii)* $\Phi_l(a, b + s) \subset \Phi_l(s' + a, b)$.

**Proof:**

(i) Let $d \in \nabla_l(a, b + s)$, then by definition, $d \leq a$ and $b + s \leq d + \tilde{a}$. The latter inequality implies $b \leq (d + \tilde{a})\backslash s = d + (\tilde{a}\backslash s)$. This implies $d \leq a \leq s' + a$, and $b \leq d + (s' + a)^{\sim}$, that is, $d \in \nabla_l(s' + a, b)$.
(ii) Let $d \in \Phi_l(a, b + s)$. It suffices to show that $d$ is minimal in $\nabla_l(s' + a, b)$. So choose $e \in E$, $e \leq d$, and $e \in \nabla_l(s' + a, b)$. From $e \leq d$ it follows that $e \leq a$. Moreover, $b \leq e + (s' + a)^{\sim} = e + \tilde{a}\backslash s$ implies $b + s \leq e + \tilde{a}$. Hence $e \in \nabla(a, b + s)$, and minimality of $d$ implies that $e = d$. This concludes the proof. □

**Lemma 3.2.** *Let $a, b \in E$, if $s \in E$ is such that $s + b$ and $s/\overline{a}$ are defined, then*

*(i)* $\nabla_r(a, s + b) \subset \nabla_r(a + s', b)$, *where* $s/\overline{a} = (a + s')^{-}$.
*(ii)* $\Phi_r(a, s + b) \subset \Phi_r(a + s', b)$.

**Proof:**

(i) Let $d \in \nabla_r(a, s + b)$, then by definition, $d \leq a$ and $s + b \leq \overline{a} + d$. The latter inequality implies $b \leq s/(\overline{a} + d) = s/\overline{a} + d = (a + s')^{-} + d$, hence $d \in \nabla_r(a + s', b)$.
(ii) Let $d \in \mathbf{\Phi}_r(a, s + b)$, we have to prove that $d$ is minimal in $\nabla_r(a + s', b)$. Let $e \leq d$, $e \in \nabla_r(a + s', b)$. This implies $e \leq d \leq a$, and $b \leq$

$(a+s')^- + e = (s/\bar{a}) + e$, from the last inequality we get $s + b \le \bar{a} + e$. Hence $e \in \Phi_r(a, s+b)$, and minimality of $d$ yields $e = d$. This concludes the proof. □

Now we are able to prove the main result in this section.

**Theorem 3.3.** *Let $E$ be a pseudoeffect algebra such that for every $a, b \in E$, $\Phi_l(a, b) \neq \emptyset$ and $\Phi_r(a, b) \neq \emptyset$. An ideal $I$ in $E$ is a Riesz ideal if and only if $I$ is closed under generalized Sasaki projections.*

**Proof:** First assume that $I$ is a Riesz idal. Let $i \in I$ and $m \in \Phi_l(a, i)$, that is, $m \le a$ and $i \le m + \tilde{a}$. Since $I$ is Riesz, there are $i_1, i_2$ in $I$ such that $i \le i_1 + i_2$ and $i_1 \le m$, $i_2 \le \tilde{a}$. This implies $i_1 \le a$, $i \le i_1 + \tilde{a}$ hence $i_1 \in \nabla_l(a, i)$. From $i_1 \le m$, and minimality of $m$ we get $m = i_1 \in I$. This proves that $I$ is closed under left generalized Sasaki projections. Now let $i \in I$, $m \in \Phi_r(a, i)$. Then $m \le a$, $i \le \bar{a} + m$. It follows that there are $i_1, i_2 \in I$ such that $i_1 \le \bar{a}$, $i_2 \le m$ and $i \le i_1 + i_2$. This entails $i_2 \le m \le a$ and $i \le \bar{a} + i_2$ that is, $i_2 \in \nabla_r(a, i)$. Since $m$ is minimal and $i_2 \le m$ we get $m = i_2 \in I$. This proves that $I$ is closed under right generalized Sasaki projections.

To prove the converse, assume that $I$ is an ideal closed under generalized Sasaki projections. Let $i \in I$ and $c, d \in E$ be such that $i \le c + d$. Put $s := d/\tilde{c}$, the equality $c + d + s = c + d + d/\tilde{c} = 1$ implies that $s$ is defined and $i + s$ is defined. Choose $h \in \Phi_l(c, i + s)$. Then $h \le c$ and by Lemma 3.1 (ii), $h \in \Phi_l(s' + c, i)$, where $(s' + c)^\sim = \tilde{c}\backslash s = \tilde{c}\backslash(d/\tilde{c}) = d$. The last equality follows from $\tilde{c}\backslash(d/\tilde{c}) + d/\tilde{c} = \tilde{c} = d + d/\tilde{c}$. So $d = (s' + c)^\sim$, and $h \in \Phi_l(\bar{d}, i)$ implies $h \in I$. Summarizing, we have $i \le h + \tilde{\bar{d}} = h + d$, $h \le c$, and $h \in I$.

Now choose $k \in \Phi_r(d, p + i)$, where $p = \bar{d}\backslash h$. The element $p$ is defined, since $h + d$ is defined and $\bar{d}\backslash h + h + d = 1$. So we have $k \le d$, $p + i \le \bar{d} + k$, which implies $i \le p/(\bar{d} + k) = p/\bar{d} + k$, where the element $p/\bar{d}$ is defined as $p \le \bar{d}$. Let $p'$ be such that $d + p' = (p/\bar{d})^-$, then Lemma 3.2 (ii) implies that $k \in \Phi_r(d + p', i)$. Hence $k \in I$ and $i \le (d + p')^- + k$. But $(d + p')^- = p/\bar{d} = (\bar{d}\backslash h)/\bar{d} = h$, so that $i \le h + k$. This proves that $I$ is Riesz. □

## 4. LATTICE ORDERED PSEUDOEFFECT ALGEBRAS

A pseudoeffect algebra $E$ is lattice ordered if for all $a, b \in E$, $a \vee b$ and $a \wedge b$ are in $E$.

**Theorem 4.1.** *Let $E$ be a lattice ordered pseudoeffect algebra. The $\Phi_l(a, b) = \{a \wedge \bar{b}/a\}$ and $\Phi_r(a, b) = \{a\backslash a \wedge \tilde{b}\}$.*

**Proof:** Let $d \in \nabla_l(a,b)$ then $d \le a$ and $b \le d + \tilde{a}$. It then follows that $a\backslash d + d + \tilde{a} = 1$, hence $d + \tilde{a} = (a\backslash d)^{\sim}$, and therefore $b \le (a\backslash d)^{\sim}$. This entails that $a\backslash d + \bar{b}$ exists, and $a\backslash d + \bar{b} \le 1$ implies $a\backslash d \le 1\backslash \bar{b} = b$. Hence maximal possible value of $a\backslash d$ is $a \wedge b$. Put $d_0 = (a \wedge \bar{b})/a$, then $d_0 \le a$ and $(a \wedge \bar{b})/a + \tilde{a} = (a \wedge \bar{b})/1 = (a \wedge \bar{b})^{\sim}$. From $a \wedge \bar{b} \le \bar{b}$ we obtain $b \le (a \wedge \bar{b})^{\sim}$, which entails that $d_0 + \tilde{a} \ge b$. This proves that $d_0 \in \nabla_l(a,b)$, and $a\backslash d_0 = a \wedge b$ entails that $d_0$ is minimal.

Let $d \in \nabla_r(a,b)$. Then $d \le a$ and $b \le \bar{a} + d$, and similarly as above, we derive that $d/a \le a \wedge \tilde{b}$. Put $d_0 = a\backslash(a \wedge \tilde{b})$. Then $d_0 \le a$ and $\bar{a} + d_0 = \bar{a} + a\backslash(a \wedge \tilde{b}) = 1\backslash(a \wedge \tilde{b}) = (a \wedge \tilde{b})^{-}$. From $a \wedge \tilde{b} \le \tilde{b}$ we obtain that $b \le (a \wedge \tilde{b})^{-} = \bar{a} + d_0$. This proves that $d_0 \in \nabla_r(a,b)$, and since $d_0/a = (a\backslash(a \wedge \tilde{b})/a = a \wedge \tilde{b}$, $d_0$ is minimal. □

We denote $\phi_l(a,b) := (a \wedge \bar{b})/a$, $\phi_r(a,b) := a\backslash(a \wedge \tilde{b})$.

**Corollary 4.2.** *An ideal $I$ in lattice ordered pseudoeffect algebra is Riesz if and only if $b \in I$ implies $\phi_l(a,b) = (a \wedge \bar{b})/a \in I$ and $\phi_r(a,b) = a\backslash(a \wedge \tilde{b}) \in I$.*

**Lemma 4.3.** *In every pseudoeffect algebra $E$ the following holds for any $a, b \in E$, with $a \le b$:*

(i) $b\backslash a = \bar{b}/\bar{a}$,
(ii) $a/b = \tilde{a}\backslash\tilde{b}$

**Proof:** Clearly, $a \le b$ implies $\bar{b} \le \bar{a}$, $\tilde{b} \le \tilde{a}$.

(i) $b\backslash a + a = b$ implies $\bar{b} + b\backslash a + a = \bar{a} + a = 1$. From this $\bar{b} + b\backslash a = \bar{a}$, hence $b\backslash a = \bar{b}/\bar{a}$.
(ii) $a + a/b + \tilde{b} = b + \tilde{b} = a + \bar{a}$ entails $a/b + \tilde{b} = \tilde{a}$, hence $a/b = \tilde{a}\backslash\tilde{b}$. □

**Lemma 4.4.** *In a lattice ordered pseudoeffect algebra $E$ the following holds for any, $a, b \in E$:*

*(i)* $(a \vee b)^{-} = \bar{a} \wedge \bar{b}$,
*(ii)* $(a \vee b)^{\sim} = \tilde{a} \wedge \tilde{b}$.

**Proof:**

(i) From $(a \vee b)^{-} \le \bar{a}, \bar{b}$ it follows that $(a \vee b)^{-} \le \bar{a} \wedge \bar{b}$. Let $d \in E$ be such that $d \le \bar{a}, \bar{b}$. Then $a, b \le d^{\sim}$, whence $(a \vee b) \le d^{\sim}$, which entails $d \le (a \vee b)^{-}$. This proves that $(a \vee b)^{-} = \bar{a} \wedge \bar{b}$.
(ii) We have $(a \vee b)^{\sim} \le a^{\sim}, b^{\sim}$. Let $d \in E$ be such that $d \le a^{\sim}, b^{\sim}$. Then $a, b \le \bar{d}$, hence $a \vee b \le \bar{d}$, and so $d \le (a \vee b)^{\sim}$. This proves $(a \vee b)^{\sim} = \tilde{a} \wedge \tilde{b}$. □

As a corollary, we obtain $(a \wedge b)^{-} = \bar{a} \vee \bar{b}$, $(a \wedge b)^{\sim} = \tilde{a} \vee \tilde{b}$.

**Corollary 4.5.** *An ideal $I$ in a lattice ordered effect algebra $E$ is Riesz if and only if for any $b \in I$, $a \in E$, we have $a/(a \vee b) \in I$ and $(a \vee b)\backslash a \in I$. In addition, a normal ideal $I$ is Riesz if and only if one of the latter conditions holds.*

**Proof:** According to Corollary 4.2, $I$ is a Riesz ideal if and only if $(a \wedge \bar{b})/a \in I$ and $a\backslash(a \wedge \bar{b}) \in I$ whenever $b \in I$. Now we have, using Lemmas 4.3 and 4.4,

$$\begin{aligned} a\backslash(a \wedge \bar{b}) &= \bar{a}/(a \wedge \bar{b})^{-} \\ &= \bar{a}/(\bar{a} \vee b). \end{aligned}$$

Similarly,

$$\begin{aligned} (a \wedge \bar{b})/a &= (a \wedge \bar{b})^{\sim}\backslash\tilde{a} \\ &= (\tilde{a} \vee b)\backslash\tilde{a}. \end{aligned}$$

Since $a$ is arbitrary, we obtain the desired statement.

If $I$ is a normal ideal, then for any $a \in E$, $a + I = I + a$. From

$$a + a/(a \vee b) = (a \vee b)\backslash a + a,$$

and normality of $I$, we have $a/(a \vee b) \in I$ iff $(a \vee b)\backslash a \in I$. □

According to Dvurečenskij and Vetterlein (2001b), a lattice ordered effect algebra is a pseudo MV-algebra iff

$$a\backslash(a \wedge b) = (a \vee b)\backslash b \tag{5}$$

equivalently, iff

$$(a \wedge b)/a = b/(a \vee b) \tag{6}$$

From this we can derive the following noncommutative analogue of the $\phi$-symmetry condition (Bennett and Foulis, 1995) for Sasaki projections in pseudo-MV algebras.

**Theorem 4.6.** *A lattice ordered effect algebra $E$ is a pseudo MV-algebra if and only if for any $a, b \in E$, $\phi_l(a, b) = \phi_r(b, a)$.*

**Proof:** By (6) and Lemma 4.4, $\phi_l(a, b) = (a \wedge \bar{b})/a = \bar{b}/(a \vee \bar{b}) = b\backslash(\tilde{a} \wedge b) = \phi_r(b, a)$. Applying (5) to $\phi_r(a, b)$ we obtain the same relation. □

In what follows we need the following observations.

**Lemma 4.7.** *Let $E$ be a pseudoeffect algebra. Let $a; b_i$, $i \in I$, be elements in $E$ such that $b := \bigvee b_i \in E$.*

*(i) If $a + b_i$ exist for all $i$, then*

$$\bigvee_i (a + b_i) = a + b. \tag{7}$$

*Similarly, if $b_i + a$ exists in $E$ for all $i$, then*

$$\bigvee_i (b_i + a) = b + a. \tag{8}$$

*(ii) If $c \leq b_i$ for all $i$, then*

$$\bigvee_i (b_i \backslash c) = b \backslash c. \tag{9}$$

*Similarly,*

$$\bigvee_i (c / b_i) = c / b. \tag{10}$$

**Proof:**

(i) For all $i$, $a + b_i \leq a + b$. Assume that $m \in E$ is such that $a + b_i \leq m$ for all $i$. Then $b_i \leq a/m \forall i$ implies $b \leq a/m$, and this yields $a + b \leq m$. Hence $a + b$ is the supremum of $a + b_i$.

(ii) $b_i \geq c \forall i$ implies $b \geq c$, and $b_i \backslash c \leq b \backslash c \forall i$. Assume that $m \in E$ is such that $b_i \backslash c \leq m \forall i$, then $b_i \backslash c = (b_i \backslash c) \wedge \overline{c} \leq m \wedge \overline{c}$. This entails $b_i \leq m \wedge \overline{c} + c \forall i$, and so $b \leq m \wedge \overline{c} + c$, which yields $b \backslash c \leq m$. Therefore $b \backslash c$ is the supremum of $b_i \backslash c$. □

**Lemma 4.8.** *Let $E$ be a lattice ordered pseudoeffect algebra, $a, b \in E$ such that $a + b = b' + a$ exists and $a \wedge b' = 0$ then $a + b = a \vee b$.*

**Proof:** By Dvurečenskij and Vetterlein (2001a), $c \backslash (a \vee b) = (c \backslash a) \wedge (c \backslash b)$ whenever $c \geq a, b$. Put $c = a + b$, then $(a + b) \backslash (a \vee b) = ((a + b) \backslash a)) \wedge ((a + b) \backslash b) = ((b' + a) \backslash a) \wedge ((a + b) \backslash b) = b' \wedge a = 0$. □

An element e of a pseudoeffect algebra $E$ is said to be *central* (Dvuečenskij, 2003) if there exists an isomorphism

$$f_e : E \to [0, e] \times [0, \tilde{e}] \tag{11}$$

such that $f_e(e) = (e, 0)$ and if $f_e(x) = (x_1, x_2)$, then $x = x_1 + x_2$ for all $x \in E$.

**Theorem 4.9.** *Let $I$ be a normal Riesz ideal in a lattice ordered pseudoeffect algebra $E$. If $p := \bigvee\{i: i \in I\}$ exists, then $p$ is a central element in $E$.*

**Proof:** Let $p = \bigvee I$. First we prove the following properties: $\forall c \in E$, $(p \vee c)\backslash c \leq p$, $\overline{p} = \tilde{p}$ and $p \wedge \overline{p} = 0$.

By Lemma 4.7 (ii), for every $c \in E$, $(p \vee c)\backslash c = \bigvee\{(a \vee c)\backslash c: a \in I\}$, and since $I$ is Riesz, $(a \vee c)\backslash c \in I$, hence $(p \vee c)\backslash c \leq p$. Similarly, $c/(p \vee c) \leq p$ for all $c \in E$.

Let $a \in I$, then $\overline{p} \wedge a \in I$ and for all $b \in I$, $\overline{p} \wedge a + b \in I$ hence $\overline{p} \wedge a + b \leq p$. By Lemma 4.7 (i) $\overline{p} \wedge a + p = \bigvee\{(\overline{p} \wedge a) + b: b \in I\}$, hence $\overline{p} \wedge a + p \leq p$. This gives $\overline{p} \wedge a = 0$. Let $a'$ be such that $\overline{p} + a = a' + \overline{p}$, since $I$ is normal, we have $a' \in I$, so that $\overline{p} \wedge a' = 0$. By Lemma 4.8, it follows that $\overline{p} + a = \overline{p} \vee a$. Similarly we prove that $a + \tilde{p} = a \vee \tilde{p}$. Therefore

$$1 = \overline{p} + p = \bigvee\{\overline{p} + a: a \in I\} = \{\overline{p} \vee a: a \in I\} = \overline{p} \vee p$$

by Lemma 4.7, hence

$$(\overline{p} \vee p)^{\sim} = p \wedge \tilde{p} = 0.$$

Similarly,

$$1 = p + \tilde{p} = \bigvee\{a + \tilde{p}: a \in I\} = \{a \vee \tilde{p}: a \in I\} = p \vee \tilde{p}$$

yields

$$(p \vee \tilde{p})^{-} = \overline{p} \wedge p = 0.$$

From

$$(p \vee c)\backslash c = \overline{p} \wedge \overline{c}/\overline{c} \leq p \wedge \overline{c}$$

(Lemma 4.4), we get

$$\overline{c} \leq \overline{c} \wedge \overline{p} + \overline{c} \wedge p.$$

Since this holds for every $c \in E$, we obtain

$$c \leq c \wedge \overline{p} + c \wedge p \tag{12}$$

for all $c \in E$.

Similarly, from

$$c/(p \vee c) = \tilde{c}\backslash(\tilde{p} \wedge \tilde{c}) \leq p \wedge \tilde{c}$$

(Lemma 4.4), we get

$$\tilde{c} \leq \tilde{c} \wedge p + \tilde{c} \wedge \tilde{p},$$

and since this holds for all $c \in E$, we obtain

$$c \leq c \wedge p + c \wedge \tilde{p} \tag{13}$$

Putting $c = \overline{p}$, we get

$$\overline{p} \leq \overline{p} \wedge p + \overline{p} \wedge \tilde{p} = \overline{p} \wedge \tilde{p},$$

and putting $c = \tilde{p}$, we get

$$\tilde{p} \leq \tilde{p} \wedge \overline{p} + \tilde{p} \wedge p = \tilde{p} \wedge \overline{p},$$

and summarizing, we have $\overline{p} = \tilde{p}$.

From (12) we have, for all $a \in E$,

$$a/(a \wedge p + a \wedge \tilde{p}) \leq a \wedge p/(a \wedge p + a \wedge \tilde{p}) = a \wedge \tilde{p} \leq \tilde{p}. \tag{14}$$

Using Lemma 4.7, we have

$$a \wedge p + a \wedge \tilde{p} \leq p + a \wedge \tilde{p} = \bigvee\{e + a \wedge \tilde{p}: e \in I\},$$

and $e + a \wedge \tilde{p} = a \wedge \tilde{p} + e'$, where by normality of $I$, $e' \in I$. Hence $a \wedge \tilde{p} \wedge e' = 0$, and by Lemma 4.8, $e + a \wedge \tilde{p} = e \vee a \wedge \tilde{p}$. This entails

$$p + a \wedge \tilde{p} = \bigvee\{e + a \wedge \tilde{p}: e \in I\} = \bigvee\{e \vee a \wedge \tilde{p}: e \in I\} = p \vee a \wedge \tilde{p},$$

and then

$$a/(a \wedge p + a \wedge \tilde{p}) \leq a/(p \vee a \wedge \tilde{p}) \leq a/(p \vee a) \leq p. \tag{15}$$

From $p \wedge \tilde{p} = 0$ and from (14) and (15) it follows that

$$a/(a \wedge p + a \wedge \tilde{p}) = 0,$$

hence for every $a \in E$,

$$a = a \wedge p + a \wedge \tilde{p}. \tag{16}$$

Similarly we prove that

$$a = a \wedge \overline{p} + a \wedge p. \tag{17}$$

Define the mapping $f_p : E \to [0, p] \times [0, \tilde{p}]$ by

$$f_p(x) = (x \wedge p, x \wedge \tilde{p}) \tag{18}$$

Clearly, $f_p(p) = (p, 0)$ and $f_p(1) = (p, \tilde{p})$. For any $(x_1, x_2)$ with $x_1 \leq p$, $x_2 \leq \tilde{p}$ put $x = x_1 + x_2$, then $x_1 + x_2 = x \wedge p + x \wedge \tilde{p}$, and $x_1 \leq x \wedge p, x_2 \leq x \wedge \tilde{p}$ implies $x_1 = x \wedge p, x_2 = x \wedge \tilde{p}$. Moreover, from (17) and (16) and from $\tilde{p} = \overline{p}$ and $p \wedge \overline{p} = 0$, we get by Lemma 4.8, that $x = x_1 \vee x_2 = x_2 + x_1$.

It follows that if $(x_1, x_2), (y_1, y_2) \in [0, p] \times [0, \tilde{p}]$ are such that $x_1 + x_2 = y_1 + y_2$, or if $x_1 + x_2 = y_2 + y_1$, then $(x_1, x_2) = (y_1, y_2)$.

From this we can derive that $f_p$ is injective and surjective.

If $x + y$ exists, then $x \wedge p + y \wedge p$ and $x \wedge \tilde{p} + y \wedge \tilde{p}$ exist, therefore $f_p(x) + f_p(y) = (x \wedge p + y \wedge p, x \wedge \tilde{p} + y \wedge \tilde{p})$, and $x \wedge p + x \wedge \tilde{p} + y \wedge p +$

$y \wedge \tilde{p} = x + y$, sice any of $x \wedge p$, $y \wedge p$ commutes with any of $x \wedge \tilde{p}$, $y \wedge \tilde{p}$. It follows that $f_p(x) + f_p(y) = f_p(x + y)$.

Conversely, if $f_p(x) + f_p(y)$ exists, then $x \wedge p + x \wedge \tilde{p} + y \wedge p + y \wedge \tilde{p}$ exists, and equals $x + y$. This proves that $f_p$ and $f_p^{-1}$ are homomorphisms, which concludes the proof. $\square$

## ACKNOWLEDGMENT

This research is supported by grant VEGA 2/3163/23.

## REFERENCES

Avallone, A. and Vitolo, P. (2003). Congruences and ideals of effects algebras. *Order*, **20**, 67–77.

Baudot, R. (2000). Non-commutative programming language NoClog. In Symposium LICS, Santa Barbara (Short Presentation), pp. 3–9.

Beltrametti, E. G. and Cassinelli, G. (1981). *The Logic of Quantum Mechanics*, Addison-Wesley, Reading, MA.

Bennett, M. K. and Foulis, D. J. (1995). Phi-symmetric effect algebras. *Foundations of Physics*, **25**, 1699–1722.

Bennett, M. K. and Foulis, D. J. (1998). A generalized Sasaki projection for effect algebras. *Tatra Mountains Mathematical Publications*, **16**, 55–66.

Chang, C. C. (1958). Algebraic analysis of many-valued logic. *Transactions of American Mathematical Society*, **88**, 467–490.

Chevalier, G. and Pulmannová, S. (2000). Some ideal lattices in partial abelian monoids and effect algebras. *Order*, **17**, 72–92.

Dvurečenskij, A. (2003). Idelas of pseudo-effect algebras and their applications. *Tatra Mountains Mathematical Publications*.

Dvurečenskij, A. (2003). Central elements and Cantor-Bernstein theorem for pseudo-effect algebras. *Journal of Australian Mathematical Society*, **74**, 121–143.

Dvurečenskij, A. and Pulmannová, S. (2000). *New Trends in Quantum Structures*, Kluwer, Dordrecht.

Dvurečenskij, A. and Vetterlein, T. (2001a). Pseudo-effect algebras I. Basic properties. *International Journal of Theoretical Physics*, **40**, 685–701.

Dvurečenskij, A. and Vetterlein, T. (2001b). Pseudo-effect algebras II. Group representations. *International Journal of Theoretical Physics*, **40**, 703–726.

Dvurečenskij, A. and Vetterlein, T. (2001c). Congruences and states on pseudo-effect algebras. *Foundations of Physics Letters*, **14**, 425–446.

Foulis, D. J. and Bennett, M. K. (1994). Effect algebras and unsharp quantum logics. *Foundations of Physics*, **24**, 1325–1346.

Georgescu, G. and Iorgulescu, A. (2001). Pseudo-MV algebras. *Multi Valued Logic*, **6**, 95–135.

Hájek, P. (submitted). Observations on non-commutative fuzzy logic.

Pták, P. and Pulmannová, S. (1981). *Orthomodular Structures as Quantum Logics*, Kluwer, Dordrecht.

Rachůnek, J. (2002). A non-commutative generalization of MV algebras. *Czechoslovak Mathematical Journal*, **52**, 255–273.

Varadarajan, V. S. (1985). *Geometry of Quantum Theory*, Springer, Heidelberg.